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2020-07-15

Blasten , E , Paivarinta , L & Sadique , S 2020 , ' Unique Determination of the Shape of a Scattering Screen from a Passive Measurement ' , Mathematics , vol. 8 , no. 7 , 1156 . https://doi.org/10.3390/math8

http://hdl.handle.net/10138/324456
https://doi.org/10.3390/math8071156

mathematics

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Unique Determination of the Shape of a Scattering Screen from a Passive Measurement

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Received: 11 June 2020; Accepted: 10 July 2020; Published: 15 July 2020

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Abstract: We consider the problem of fixed frequency acoustic scattering from a sound-soft flat screen. More precisely, the obstacle is restricted to a two-dimensional plane and interacting with an arbitrary incident wave, it scatters acoustic waves to three-dimensional space. The model is particularly relevant in the study and design of reflecting sonars and antennas, cases where one cannot assume that the incident wave is a plane wave. Our main result is that given the plane where the screen is located, the far-field pattern produced by any single arbitrary incident wave determines the exact shape of the screen, as long as it is not antisymmetric with respect to the plane. This holds even for screens whose shape is an arbitrary simply connected smooth domain. This is in contrast to earlier work where the incident wave had to be a plane wave, or more recent work where only polygonal scatterers are determined.

Keywords: inverse scattering; screen; uniqueness; single measurement; passive measurement

MSC: 35R30; 35P25; 35A02

1. Introduction

1.1. Antennas

The motivation for the study of wave scattering from thin and large objects lies in the antenna theory. The starting point for this was when the Prussian Academy announced an open competition about who could be the first to show the existence or non-existence of electromagnetic (EM) waves in 1879. The existence of these waves were predicted 15 years earlier by the mathematical theory of James Clerk Maxwell [1]. The competition was won in 1882 by young Heinrich Hertz, in favor of Maxwell's theory. He did this by constructing a dipole antenna radiating EM waves which he could measure. It is needless to mention the importance which this experiment together with Maxwell's theory has had for modern society. Hertz's antenna consisted of two identical perfectly conducting planar bodies, in his case squares, which create radiating EM waves. Since, by reciprocity, radiating antennas are identical to receiving antennas, the theory of antennas is closely connected to EM scattering and inverse scattering theory.

A key question in antenna design for scientific radio arrays is how to choose the antenna topology so that its impedance and radiation patterns are frequency independent (FI) over a wide range of frequencies and, simultaneously, the radiation pattern supports beamforming. Well-known examples of FI antennas include log-periodic, log-spiral, and UHF fractal antennas on high-frequencies. While proven good for extremely wide band work, these are heavy and complicated structures and thus not cost-efficient for extremely large arrays.

Instead of relying on traditional antenna forms, we aim to derive general principles for designing antennas with frequency independent characteristics. A major step in such a design strategy is to solve the inverse scattering problem: given an input–output pair of waves, which antenna shape produces it? The input is a given incident wave, and the output is the far-field pattern produced by the antenna. The path to antenna design is a long one, so in this paper we study the technically easier acoustic scattering problem.

In acoustics, scattering surfaces or screens are not called antennas but sonars. Traditionally sonars are classified into active and passive sonars, depending on whether they act as a sound source or receiver. We consider acoustic scattering from screens, something which lies between these two extremes. It is more correct to call these screens passive sonars as they do not have an energy source, but they are active in the sense that their effect on the sound pattern is significant. In general the nomenclature "sonar" refers to probing using an active and passive sonar. Our research is rather in the domain of acoustic design. The mathematical question of finding a screen that scatters a given incident wave into a particular far-field has applications like the following, for example: how to reduce echo in an office space? How to direct acoustic vibrations or reduce them? Of course, it also answers the probing question: can we determine the shape and location of a passive sonar by how it reflects sound? These are complex questions, only one part of which we are going to solve, namely that a single input–output pair of sound waves uniquely determines the shape of a flat acoustic screen.

1.2. Mathematical Background

The problem of inverse scattering with reduced measurement data has gained a lot of interest lately. Traditionally determining a scatterer from far-field measurements requires sending all possible incident waves and recording the corresponding far-field patterns. The method of using complex geometrical optics solutions infinitely many far-field measurements in the fixed frequency setting was pioneered by Sylvester and Uhlmann in [2], and was the first method for uniquely determining an arbitrary smooth enough scattering potential by far-field measurements. The field has grown extremely fast since then, almost to the point of saturation, and we will only point the reader towards the surveys in [3] for references up to 2003, which gives a good picture of the situation except for scattering in two dimensions, which was solved by Bukhgeim [4] in 2007 and improved by several authors, e.g., [5–9].

In many applications the scatterer is impenetrable, or we are only interested in its shape or location. The shape determination problem is known as Schiffer's problem in the literature [10]. M. Schiffer showed that a sound-soft obstacle (with non-empty interior) can be uniquely determined by infinitely many far-field patterns. The proof appeared as a private communication in the monograph by Lax and Phillips [11]. Linear sampling [12] and factorization [13] methods were developed and they are very well suited for shape determination, also from the numerical point of view. These were applied in the context of curved screens in acoustic [14] and electromagnetic [15] scattering to determine the shape and location of the screen, also numerically. However, these methods require the full use of infinitely many far-field patterns, except for a case of interest in [14] to which we will return later in more detail.

There was still much to improve: counting dimensions shows that a single far-field (a mapping $\mathbb{S}^{n-1} \to \mathbb{C}$) should be enough to determine the shape (a manifold of dimension $n-1$). Colton and Sleeman reduced the requirements to finitely many far-field patterns [16]. It is widely conjectured that the uniqueness for Schiffer's problem follows from a single far-field pattern [10,17], and the situation for a general shape is wide open. This brings in the current results. Various authors proved at roughly the same time in the recent past that polyhedral sound-soft obstacles are uniquely determined by a single far-field pattern in various settings [18–24]. Part of the results above apply for screens as long as the screen is polygonal. A special case in [14] gives the unique determination of a flat screen by a single incident plane-wave measurement. Their proof requires that the incident wave has non-vanishing properties everywhere on the plane where the screen is located—an issue that we remedy completely. So far there is no proof for the unique determination of an obstacle's shape by one far-field pattern without restrictive a priori assumptions. The results in [25] come very close: the obstacle can be any

Lipschitz domain as long as its boundary is not an analytic manifold. It does not allow screens, which is our focus.

An alternative approach to unique determination which has gained interest recently, is to consider what can be determined with less data, e.g., one measurement, in the setting of penetrable scatterers which were usually treated with various methods based on the Sylvester—Uhlmann [2] or Bukhgeim [4] papers. Much of the recent work taking this point of view uses unique continuation results and precise analysis on the behavior of Fourier transforms of the characteristic functions of various shapes [26–32]. A very interesting point of view is determining the so-called convex scattering support [33,34] by one far-field measurement. Again, none of the above are applicable to screens per se.

Our work in this paper shows that given the far-field caused by any single given incident wave scattering off a smooth flat screen, the latter's shape is determined uniquely. Our methods are based on ideas which are partly motivated by the study of certain integral operators in [35,36]. As in [14], we first show that the far-field is the restriction to a ball of radius k (the wavenumber) of the two-dimensional Fourier transform of a function supported on the screen. Next, since the incident wave might vanish on part of the screen, we show that the shape of the screen is exactly the support of that function. This latter part involves a delicate analysis of the Taylor coefficients of the scattered wave at the screen, but it leads to our main theorem: that Schiffer's problem is uniquely solvable for flat screens on a plane in three dimensions, for any incident wave that causes scattering.

Let us discuss the significance of our result, with focus especially on our improvements over [14]: that any incident field is allowed. We will start with the mathematical challenges. Unlike for infinite measurements inverse problems such as [2,4], properties of the incident wave affect greatly the solvability of single measurement inverse problems. Complex plane waves make things technically simpler in many scattering problems because of their explicit form and non-vanishing everywhere. This often reduces the non-linear inverse scattering problem to the linear inverse source problem after a suitable interpretation, or avoids other challenges, as can be seen by comparing [26,27,31] to [28–30]. Furthermore, in situations involving scattering from multiple objects, the total incident field impinging on a given component is the sum of the original incident field and the fields scattered by the other components. This is relevant when one wishes to uniquely determine a screen where space contains other scatterers that are known. On the other hand, from the applied point of view, solving the inverse problems for any given incident field enables *passive measurements*. This means that even if we do not have control over the incident wave, or cannot afford to control it, the shape of the scatterer can be uniquely determined. This is both good and bad. It means that the flat screen design problem of finding its shape such that it scatters one given incident wave into a given far-field has no more than a unique solution. On the other hand it shows the impossibility of more complex input–output systems. One cannot require it to scatter two or more incident waves into their corresponding far-fields in general. The first incident wave and far-field pair already determines the shape.

Lastly, we remark that inverse scattering for screens has still many open problems. Current solutions require that the screen have at least a differentiable boundary, something which arises from the way that the direct scattering problem has been shown solvable in [37] and other sources. To bring forward the range characterization condition from [14] to the situations of let us say Herglotz incident waves, one would need to solve a deconvolution problem. A more difficult and certainly more interesting question mathematically and from the point of view of applications, is the unique determination of the shape of a curved screen from one measurement, passive or fully controlled. The problem is solved for infinitely many measurements in [14], but counting dimensions suggests that it should be solvable with one measurement.

1.3. Definitions and Theorems

Let us go forward to the mathematics. We start by defining what we mean by a screen and the scattering problem from screens. Then we state our three main theorems. They give representation formulas for the scattered wave, the far-field pattern, and the unique solvability of Schiffer's problem for determining the shape of a scattering screen using a single incident wave. In Section 2 we prove the representation formulas, and then in Section 3 we solve the inverse problem.

We consider the scattering of a two-dimensional sound-soft and flat obstacle Ω in three-dimensional space. We will assume that Ω is an open subset of $\mathbb{R}^2 \times \{0\}$.

Definition 1. *We call a set $\Omega \subset \mathbb{R}^3$ a screen, if $\Omega = \Omega_0 \times \{0\}$ for some simply connected bounded domain $\Omega_0 \subset \mathbb{R}^2$ whose boundary is smooth, and which we call its shape.*

The scattering of acoustic waves by Ω leads to the study of the Helmholtz equation $(\Delta + k^2)u = 0$ where the wave number k is given by the positive constant $k = \omega/c$ where c is the constant speed of sound in the background fluid (air, water, etc.) and ω is the angular frequency of the wave. The pressure of the total wave vanishes on the boundary of a sound-soft obstacle, and the total wave is a sum of the incident and scattered waves. This leads to the following set of partial differential equations.

Definition 2. *We define the direct scattering problem for a screen Ω as follows. Given an incident wave u_i satisfying $(\Delta + k^2)u_i = 0$ in $\mathbb{R}^3$ and a screen Ω, the direct scattering problem has a solution if there is $u_s \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{\Omega})$ that satisfies the following conditions*

$$(\Delta + k^2)u_s = 0, \qquad \mathbb{R}^3 \setminus \overline{\Omega}, \tag{1}$$

$$u_i(x) + u_s(x) = 0, \qquad x \in \Omega, \tag{2}$$

$$r\Big(\frac{\partial}{\partial r} - ik\Big)u_s = 0, \qquad r \to \infty, \tag{3}$$

where $r = |x|$ and the limit is uniform over all directions $\hat{x} = x/r \in \mathbb{S}^2$ as $r \to \infty$.

There are a few things above that we should clarify. By $H^1_{loc}(\mathbb{R}^3 \setminus \overline{\Omega})$ we mean the set of distributions ψ on $\mathbb{R}^3 \setminus \overline{\Omega}$ for which $\psi_{|U} \in H^1(U)$ for any bounded convex open set $U \subset \mathbb{R}^3 \setminus \overline{\Omega}$. Secondly, since strictly speaking u_s is not defined on Ω, by (2) we mean that the Sovolev trace of u_s both from above ($x_3 > 0$) and below ($x_3 < 0$) coincides, and is equal to $-u_i$ on Ω.

We shall start by showing a representation formula, the one in (4), for solutions u_s of the direct scattering problem for the screen. This is mainly done so that the reader would get a better intuition about this type of problems and to fix notation and function spaces clearly. This formula is well known, and it gives a unique solution to the direct problem [37]. After that we will show that the far-field, defined below, corresponding to a single given non-trivial incident wave uniquely determines the screen Ω. This type of theorem was shown in [14] on the condition that the incident wave does not vanish on the plane $\mathbb{R}^2 \times \{0\}$. To get rid of this assumption, we must show Lemma 7. We remark that the far-field pattern exists and is unique for each u_s satisfying the following assumptions. See [10] for reference.

Definition 3. *Let u_s satisfy the Sommerfeld radiation condition of (3) and the Helmholtz equation $(\Delta + k^2)u_s = 0$ outside a ball $B \subset \mathbb{R}^3$. We say that u_s^∞ is the far-field of u_s if*

$$u_s(x) = \frac{e^{ik|x|}}{|x|}\left(u_s^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|}\right)\right)$$

uniformly over $\hat{x}$ as $x \to \infty$.

We define some notation which will be useful throughout the whole text.

- $x, y, \ldots$ represent variables in $\mathbb{R}^3$, and we associate to them various projections described below.
- $x', y', \ldots$ mean variables in $\mathbb{R}^2$ or projections to $\mathbb{R}^2$. For example if $x = (1,2,3) \in \mathbb{R}^3$ then in that context $x' = (1,2) \in \mathbb{R}^2$, but we could have dy' in an integral over a subset of $\mathbb{R}^2$ without having to define the variable y separately.
- $x^0, y^0, \ldots$ denote lifts to $\mathbb{R}^3$, meaning $x^0 = (x', 0)$. For example if $x' = (-1,-2)$ then $x^0 = (-1,-2,0)$. This notation can also be used as a projection $\mathbb{R}^3 \to \mathbb{R}^2 \times \{0\}$. So, if $x = (1,2,3)$ then $x^0 = (1,2,0)$. Essentially $x'^0 = (x')^0 = x^0$ and $x^{0'} = (x^0)' = x'$ but we do not use this combined notation explicitly.
- Φ is reserved for the fundamental solution to $(\Delta + k^2)$, defined in Lemma 2.
- u^+, u^- mean the function u restricted to $\mathbb{R}^2 \times \mathbb{R}_+$ and $\mathbb{R}^2 \times \mathbb{R}_-$, respectively. If their variable is in $\mathbb{R}^2 \times \{0\}$ then they are the two-sided limits (traces) as $x_3 \to 0$. We often use $\partial_3 u^+$ and $\partial_3 u^-$. These are simply the derivatives in the x_3-direction of u^+ and u^-, respectively. Often this is evaluated on $\mathbb{R}^2 \times \{0\}$ where it then denotes the one-sided derivative, i.e., the trace of $\partial_3 u^{\pm}$.
- $\widetilde{H}^{-1/2}(\Omega_0)$: this is the set of $H^{-1/2}(\mathbb{R}^2)$ distributions whose support is contained in $\overline{\Omega_0}$, where we recall that Ω_0 signifies the shape of a screen Ω.

Let us discuss the direct scattering problem (1)–(3) first. In Section 2, Lemma 4, we will show the well-known representation formula

$$u_s(x) = \int_{\mathbb{R}^2} \Phi(x, y^0)\rho(y')dy' \tag{4}$$

for all $x \in \mathbb{R}^3 \setminus \overline{\Omega}$, where

$$\rho(y') = \partial_3 u_s^+(y^0) - \partial_3 u_s^-(y^0) \tag{5}$$

is an element of $\widetilde{H}^{-1/2}(\Omega_0)$ and the integral in (4) is interpreted as a distribution pairing between ρ and the smooth test function Φ restricted to the screen. Taking the trace $x \to \Omega$ in (4) and recalling that $u_s = -u_i$ on Ω in the sense of traces, (2), we get

$$u_i(x) = -\int_{\mathbb{R}^2} \Phi(x, y^0)\rho(y')dy'. \tag{6}$$

Now, for any candidate solution $u_s \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{\Omega})$, it solves the direct problem (1)–(3) if and only if ρ, as defined above, is in $\widetilde{H}^{-1/2}(\Omega_0)$ and is the solution to (6). More precisely, given ρ solving the integral equation, we can define u_s by (4), and it would solve the direct scattering problem. This was shown in Theorem 2.5 in [37]. Theorem 2.7 in the same source proves that (6) has a unique solution $\rho \in \widetilde{H}^{-1/2}(\Omega_0)$ given any $u_i \in H^{1/2}(\Omega_0)$.

Our main contributions are the following. The first of which is the familiar far-field representation derived from (4) if ρ is a function. We generalize is to distributions in $H^{-1/2}(\mathbb{R}^2)$. This is required for consistency of the function spaces involved. This detail has not been stated explicitly in earlier work involving scattering from screens.

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ be a screen and u_s satisfy the direct scattering problem for some incident field u_i and screen Ω. Then its far-field has the representation*

$$u_s^\infty(\hat{x}) = \frac{1}{4\pi}\left\langle \left(\partial_3 u_s^+ - \partial_3 u^-\right)(y^0), e^{-ik\hat{x}\cdot y^0} \right\rangle_{y'} \tag{7}$$

for $\hat{x} \in \mathbb{S}^2$. If $\partial_3 u_s^+ - \partial_3 u_s^-$ is integrable on Ω, this formula is equivalent to

$$u_s^\infty(\hat{x}) = \frac{1}{4\pi}\int_{\mathbb{R}^2} e^{-ik\hat{x}\cdot y^0}\left(\partial_3 u_s^+ - \partial_3 u^-\right)(y^0)dy'.$$

Our main theorem shows that even with an unoptimal incident wave, the scattering caused by it from flat screens determines the shape uniquely.

Theorem 2. *Let $\Omega, \tilde{\Omega} \subset \mathbb{R}^3$ be screens and $k \in \mathbb{R}_+$. Let u_i be an incident wave and $u_s, \tilde{u}_s$ be scattered waves that satisfy the direct scattering problem for screens $\Omega, \tilde{\Omega}$, respectively.*

If u_i is not antisymmetric with respect to $\mathbb{R}^2 \times \{0\}$ and $u_s^\infty = \tilde{u}_s^\infty$, then $\Omega = \tilde{\Omega}$. If it is antisymmetric then $u_s^\infty = \tilde{u}_s^\infty = 0$ for any screens $\Omega, \tilde{\Omega}$.

2. Representation Theorems

In this section, we will prove that solutions to the direct scattering problem satisfy (4). In essence we present the well-known but very condensed argument of [37] in more detail for the convenience of the readers. We will start with representation formulas for smooth functions and then approximate the H^1-smooth u_s. At the end of the section we will prove Theorem 1.

Lemma 1. *Let $D \subset \mathbb{R}^3$ be a bounded domain whose boundary is piecewise of class C^1 and let ν denote the unit normal vector to the boundary ∂D directed to the exterior of D. Then, for $u, v \in C^2(\overline{D})$ we have Green's second formula*

$$\int_D (v\Delta u - u\Delta v)dx = \int_{\partial D} \left(\frac{\partial u}{\partial \nu} v - u\frac{\partial v}{\partial \nu}\right) ds \tag{8}$$

where ds is the surface measure of ∂D.

Proof. Theorem 3 in Appendix C.2 of [38]. □

Lemma 2. *Let $D \subset \mathbb{R}^3$ be a bounded domain whose boundary is piecewise of class C^1 and $k \in \mathbb{R}_+$. Let*

$$\Phi(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$$

for $x, y \in \mathbb{R}^3$, $x \neq y$. Then for any $\varphi \in C^2(\overline{D})$ and $x \in \mathbb{R}^3 \setminus \partial D$ we have

$$\begin{aligned}\int_D \Phi(x,y)(\Delta + k^2)\varphi(y)dy = &\int_{\partial D} \big(\Phi(x,y)\partial_\nu\varphi(y) - \varphi(y)\partial_\nu\Phi(x,y)\big)ds(y) \\ &+ \begin{cases} 0, & x \in \mathbb{R}^3 \setminus \overline{D}, \\ -\varphi(x), & x \in D. \end{cases}\end{aligned} \tag{9}$$

Proof. We have $(\Delta + k^2)\varphi$ bounded and $y \mapsto \Phi(x,y)$ integrable for any x, so

$$\int_D \Phi(x,y)(\Delta + k^2)\varphi(y)dy = \lim_{r\to 0} \int_{D\setminus B(x,r)} \Phi(x,y)(\Delta + k^2)\varphi(y)dy.$$

Green's second formula, given in (8), applied to the integral on the right gives

$$\begin{aligned}\ldots = &\int_{D\setminus B(x,r)} (\Delta + k^2)\Phi(x,y)\varphi(y)dy \\ &+ \int_{S(x,r)\cap\overline{D}} \big(\Phi(x,y)\partial_\nu\varphi(y) - \varphi(y)\partial_\nu\Phi(x,y)\big)ds(y) \\ &+ \int_{\partial D\setminus\overline{B}(x,r)} \big(\Phi(x,y)\partial_\nu\varphi(y) - \varphi(y)\partial_\nu\Phi(x,y)\big)ds(y).\end{aligned}$$

The first integral here vanishes because $(\Delta_y + k^2)\Phi(x,y) = 0$ when $y \neq x$.

The integral over $\partial D \setminus \overline{B}(x,r)$ gives the second term in the claim when $r \to 0$ because $\Phi, \partial\Phi$ are integrable since $x \notin \partial D$. Let us estimate the first term in the first boundary integral. We have

$$\int_{S(x,r)\cap\overline{D}} \Phi(x,y)\partial_\nu \varphi(y) ds(y) = \int_{S(x,r)\cap\overline{D}} \frac{e^{ikr}}{4\pi r} \partial_\nu(y) ds(y)$$

and by the ML-inequality we have

$$\left| \int_{S(x,r)\cap\overline{D}} \Phi(x,y)\partial_\nu \varphi(y) ds(y) \right| \leq \frac{1}{4\pi r} \sup_{y\in S(x,r)\cap\overline{D}} |\nabla\varphi(y)| 4\pi r^2 \to 0$$

as $r \to 0$ because $|\nabla\varphi|$ has a uniform bound in $\overline{D}$. In the last integral we have $\partial_n u\Phi(x,y) = -\partial_r\big(e^{ikr}/(4\pi r)\big) = -ike^{ikr}/(4\pi r) + e^{ikr}/(4\pi r^2)$. The integral involving $ike^{ikr}/(4\pi r)$ can be estimated as above to conclude that it vanishes when $r \to 0$. The remaining integral is

$$-\frac{e^{ikr}}{4\pi r^2} \int_{S(x,r)\cap\overline{D}} \varphi(y) ds(y) = -\frac{e^{ikr}}{4\pi r^2} \int_{S(x,r)\cap\overline{D}} \big(\varphi(y) - \varphi(x)\big) ds(y) - \frac{e^{ikr}}{4\pi r^2} \varphi(x) s\big(S(x,r)\cap\overline{D}\big).$$

We have $|\varphi(y) - \varphi(x)| \leq \sup_{\xi\in\overline{D}} |\nabla\varphi(\xi)||x-y|$ so the absolute value of the first integral above can be estimated as

$$\ldots \leq \frac{\sup|\nabla\varphi|}{4\pi r^2} \int_{S(x,r)\cap\overline{D}} |x-y| dy = \frac{\sup|\nabla\varphi|}{4\pi r^2} rs\big(S(x,r)\cap\overline{D}\big) \to 0$$

as $r \to 0$. The form of the remaining term implies the claim in each of the cases $x \in D$, $x \in \mathbb{R}^3 \setminus \overline{D}$. $\square$

Lemma 3. *Let $D \subset \mathbb{R}^3$ be a bounded domain with smooth boundary and $k \in \mathbb{R}_+$. Let $u_s \in H^1(D)$ with $(\Delta + k^2)u_s \in L^2(D)$. Then*

$$u_s(x) = -\int_D \Phi(x,y)(\Delta + k^2)u_s(y) dy + \int_{\partial D} \big(\Phi(x,y)\partial_\nu u_s(y) - u_s(y)\partial_\nu\Phi(x,y)\big) ds(y) \tag{10}$$

for $x \in D$ in the distribution sense. For $x \in \mathbb{R}^3 \setminus \overline{D}$ we have

$$0 = -\int_D \Phi(x,y)(\Delta + k^2)u_s(y) dy + \int_{\partial D} \big(\Phi(x,y)\partial_\nu u_s(y) - u_s(y)\partial_\nu\Phi(x,y)\big) ds(y) \tag{11}$$

in the distribution sense. Here the boundary integrals involving $\partial_\nu u_s$ are to be interpreted as distribution pairings between a $H^{-1/2}(\partial D)$ function and a test function.

Proof. We will prove only the first case, namely $x \in D$. The second one follows similarly. Let $(\varphi_j)_{j=0}^\infty$ be a sequence of smooth functions defined on $\overline{D}$ such that

$$\|u_s - \varphi_j\|_{H^1(D)} + \left\|(\Delta + k^2)(u_s - \varphi_j)\right\|_{L^2(D)} \to 0$$

as $j \to \infty$. Such a sequence exists, for example by convolving u_s with a mollifier ψ_ε, as in $\varphi_j = (u_s * \psi_{1/j})_{|\overline{D}}$.

We have $\Phi(x,y) = \Psi(x-y)$ for $\Psi(z) = \exp(ik|z|)/(4\pi|z|)$ which is locally integrable in $\mathbb{R}^3$. Hence the first term in the right-hand side of (10), equal to $\Psi * (\Delta + k^2)u_s$, can be approximated by $\Psi * (\Delta + k^2)\varphi_j$ in the $L^2(D)$-sense.

For any $x \in D$ the second integral in (10) is well defined because $y \mapsto \Phi(x,y)$ and $y \mapsto \partial_\nu\Phi(x,y)$ are smooth on the smooth manifold ∂D. Moreover, the x-dependence is smooth, so the mapping

$$u_s \mapsto \int_{\partial D} u_s(y)\partial_\nu\Phi(x,y) ds(y)$$

is bounded $H^1(D) \to H^{1/2}(\partial D) \to C^0(D)$ and similarly

$$u_s \mapsto \int_{\partial D} \Phi(x,y)\partial_\nu u_s(y))ds(y)$$

is bounded $H^1(D) \to H^{-1/2}(\partial D) \to C^0(D)$ when the integral is interpreted as a distribution pairing between a $H^{-1/2}(\partial D)$-function and a test function. The continuity does not necessarily hold up to the boundary. Because $\varphi_j \to u_s$ in $H^1(D)$ and the trace operators map $\mathrm{Tr} : H^1(D) \to H^{1/2}/D)$, $\partial_\nu : H^1(D) \to H^{-1/2}(\partial D)$, so the boundary integrals with u_s replaced by φ_j converge to the corresponding ones in $C^0(D)$, namely uniformly over compact subsets of D.

In conclusion, for a test function $\psi \in C_0^\infty(D)$ we have

$$\begin{aligned}
\langle u_s, \psi\rangle &= \lim_{j\to\infty} \langle \varphi_j, \psi\rangle \\
&= \lim_{j\to\infty} \left\langle -\int_D \Phi(x,y)(\Delta+k^2)\varphi_j(y)dy + \int_{\partial D} \big(\Phi(x,y)\partial_\nu\varphi_j(y) - \varphi_j(y)\partial_\nu\Phi(x,y)\big)ds(y), \psi(x)\right\rangle_x \\
&= \left\langle -\int_D \Phi(x,y)(\Delta+k^2)u_s(y)dy + \int_{\partial D} \big(\Phi(x,y)\partial_\nu u_s(y) - u_s(y)\partial_\nu\Phi(x,y)\big)ds(y), \psi(x)\right\rangle_x
\end{aligned}$$

so the equality holds in $\mathscr{D}'(D)$. □

Lemma 4. *Let $\Omega \subset \mathbb{R}^3$ be a screen, $k \in \mathbb{R}_+$ and Φ the fundamental solution from Lemma 2. Let $u_s \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{\Omega})$. If $(\Delta+k^2)u_s = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$ and it satisfies the Sommerfeld radiation condition, then*

$$u_s(x) = \int_{\mathbb{R}^2} \Phi(x,y^0)(\partial_3 u_s^+ - \partial_3 u_s^-)(y^0)dy' \tag{12}$$

for $x \in \mathbb{R}^3 \setminus \overline{\Omega}$. Also $y' \mapsto (\partial_3 u_s^+ - \partial_3 u_s^-)(y^0)$ is in $\widetilde{H}^{-1/2}(\Omega_0)$, and more precisely the integral above represents the distribution pairing of a $\widetilde{H}^{-1/2}(\Omega_0)$-function with the smooth test function Φ restricted to $\mathbb{R}^2 \times \{0\}$ on the y-variable.

Proof. Fix $x \in \mathbb{R}^3 \setminus \overline{\Omega}$. Let $D \subset \mathbb{R}^3$ be a bounded domain with smooth boundary for which $x \in D$ and $\Omega \subset \partial D$ and furthermore we want this set to be *on top* of Ω, namely that its boundary normal pointing to the interior at Ω is e_3 and not $-e_3$. Let $R > \sup_{z\in D}|x-z|$. We will use the formulas of Lemma 3 on D, which has Ω on its boundary, and $B(x,R) \setminus \overline{D}$.

First note that since $(\Delta+k^2)u_s = 0$ only the boundary integrals on the right-hand sides of (10) and (11) remain. We will see the first integral as is, namely

$$u_s(x) = \int_{\partial D} \big(\Phi(x,y)\partial_\nu^D u_s(y) - u_s(y)\partial_\nu^D \Phi(x,y)\big)ds(y), \tag{13}$$

where we denote by ∂_ν^D the internal boundary normal derivative of D, applied to functions on D. We will have the integrals in (11) to be over the set $B(x,R) \setminus \overline{D}$. The boundary of this set is $S(x,r) \cup \partial D$, and the boundary normal pointing to its interior is $-e_3$ on $\Omega \subset \partial(B(x,R) \setminus \overline{D})$. We will split the boundary integral accordingly, and in the integral over ∂D we denote by $\partial_\nu^{D_c}$ the *external* boundary normal derivative applied to function on $B(x,R) \setminus \overline{D}$. In conclusion (11) becomes

$$\begin{aligned}
0 = &\int_{S(x,R)} \big(\Phi(x,y)\partial_\nu u_s(y) - u_s(y)\partial_\nu\Phi(x,y)\big)ds(y) \\
&+ \int_{\partial D} \big(\Phi(x,y)(-\partial_\nu^{D_c})u_s(y) - u_s(y)(-\partial_\nu^{D_c})\Phi(x,y)\big)ds(y).
\end{aligned} \tag{14}$$

Finally, by interior elliptic regularity we see that u_s is continuous (in fact real analytic) in some neighborhood of x. Also, because x is outside of ∂D and $S(x,R)$, the individual boundary integrals above are continuous. Hence the equality in the sense of distributions is in fact a pointwise equality for continuous functions. In other words, both of (13) and (14) hold as continuous functions. We still remind that the integrals involving $\partial_\nu u_s$ represent distribution pairings for an element of $H^{-1/2}(\partial D)$ with that of a smooth Φ.

Let us add (13) and (14). By smoothness, $\partial_\nu^D \Phi = \partial_\nu^{D_c}\Phi$. Please note that two-sided Sobolev traces of H^1-functions yield identical results, so the integrals of $u_s\partial_\nu^D\Phi$ and $u_s\partial_\nu^{D_c}\Phi$ in (13) and (14) cancel out. The sum then gives

$$u_s(x) = \int_{S(x,R)} \big(\Phi(x,y)\partial_\nu u_s(y) - u_s(y)\partial_\nu\Phi(x,y)\big)ds(y) + \int_{\partial D} \Phi(x,y)\big(\partial_\nu^D u_s - \partial_\nu^{D_c} u_s\big)(y)ds(y). \quad (15)$$

Please note that as $R\to\infty$ the first integral in (15) vanishes because u_s satisfies the Sommerfeld radiation condition. Also, u_s is C^1 outside of $\overline{\Omega}$ by elliptic interior regularity, so the second integral's integrand is zero when $y\notin\overline{\Omega}$. Thus, letting $R\to\infty$ gives

$$u_s(x) = \int_\Omega \Phi(x,y)\big(\partial_\nu^D u_s - \partial_\nu^{D_c} u_s\big)(y)dy$$

which implies the claim as $\partial_\nu^D u_s = \partial_3 u_s^+$ and $\partial_\nu^{D_c} u_s = \partial_3 u_s^-$ on $\Omega\subset\mathbb{R}^2\times\{0\}$. Furthermore, as above, since u_s is C^1 outside of $\overline{\Omega}$, we see that $\partial_3 u_s^+ - \partial_3 u_s^- = 0$ outside of $\overline{\Omega}$, so the integrand in the statement is in $\widetilde{H}^{-1/2}(\Omega_0)$, as claimed. $\square$

With the proposition above, we are almost ready to prove the formula for the far-field of a wave scattered by a screen, Theorem 1. But first let us prove a lemma.

Lemma 5. *Let $k\in\mathbb{R}_+$ and $K\subset\mathbb{R}^3$ be a non-empty compact set. Then*

$$\lim_{r\to\infty}\sup_{|x|=r}\sup_{y\in K}|x|\left|\partial_y^\alpha\left(\frac{e^{ik|x-y|}}{|x-y|} - \frac{e^{ik|x|}}{|x|}e^{-ik\hat{x}\cdot y}\right)\right| = 0$$

for any multi-index $\alpha\in\mathbb{N}^3$ with $|\alpha|\le 1$. Recall that $\hat{x} = x/|x|$.

Proof. The case of $|\alpha|=0$ is well known, see for example the proof of Theorem 2.5 in [10]. For $|\alpha|=1$ we will instead show the equivalent statement with ∂_y^α replaced by ∇_y. Recall the following differentiation rules

- $\nabla_y|x-y|^s = -s\frac{x-y}{|x-y|}|x-y|^{s-1}$ for all $s\in\mathbb{R}$,
- $\nabla_y e^{ik|x-y|} = -ik\frac{x-y}{|x-y|}e^{ik|x-y|}$, and
- $\nabla_y e^{-ik\hat{x}\cdot y} = -ik\hat{x}e^{-ik\hat{x}\cdot y}$.

These imply that

$$\begin{aligned}\nabla_y\left(\frac{e^{ik|x-y|}}{|x-y|} - \frac{e^{ik|x|}}{|x|}e^{-ik\hat{x}\cdot y}\right) &= -ik\frac{x-y}{|x-y|}\frac{e^{ik|x-y|}}{|x-y|} + \frac{x-y}{|x-y|}\frac{e^{ik|x-y|}}{|x-y|^2} + ik\hat{x}\frac{e^{ik|x|}}{|x|}e^{-ik\hat{x}\cdot y}\\ &= -ik\left(\frac{x-y}{|x-y|} - \hat{x}\right)\frac{e^{ik|x-y|}}{|x-y|} - ik\hat{x}\left(\frac{e^{ik|x-y|}}{|x-y|} - \frac{e^{ik|x|}}{|x|}e^{-ik\hat{x}\cdot y}\right) + \frac{x-y}{|x-y|}\frac{e^{ik|x-y|}}{|x-y|^2}.\end{aligned}$$

Let us consider the three types of terms above. To prove the estimate, let us take the absolute value and multiply by $|x|$. The last one gives

$$|x|\left|\frac{x-y}{|x-y|}\frac{e^{ik|x-y|}}{|x-y|^2}\right| = \frac{|x|}{|x-y|^2}\to 0$$

uniformly as $y \in K$, $|x| = r$ and $r \to \infty$. The first term gives

$$|x|\left|-ik\left(\frac{x-y}{|x-y|}-\hat{x}\right)\frac{e^{ik|x-y|}}{|x-y|}\right| = k\frac{|x|}{|x-y|}\left|\frac{x-y}{|x-y|}-\frac{x}{|x|}\right|$$

where can still estimate

$$\left|\frac{x-y}{|x-y|}-\frac{x}{|x|}\right| = \left|\frac{x-y}{|x-y|}\frac{|x|-|x-y|}{|x|}-\frac{y}{|x|}\right| \leq \frac{||x|-|x-y||}{|x|}+\frac{|y|}{|x|} \leq 2\frac{|y|}{|x|}$$

because $||x|-|x-y|| \leq |y|$ by the triangle inequality. Thus, the first term also tends to zero uniformly as $r \to \infty$. Lastly, the second one is estimated as

$$|x|\left|-ik\hat{x}\left(\frac{e^{ik|x-y|}}{|x-y|}-\frac{e^{ik|x|}}{|x|}e^{-ik\hat{x}\cdot y}\right)\right| = k|x|\left|\frac{e^{ik|x-y|}}{|x-y|}-\frac{e^{ik|x|}}{|x|}e^{-ik\hat{x}\cdot y}\right|$$

which tends to zero uniformly because this is the case $|\alpha| = 0$ proven at the beginning of this proof. □

Proof of Theorem 1. By the definition of the far-field there is a finite constant $C > 0$ independent of x such that

$$\left|u_\infty(\hat{x}) - |x|e^{-ik|x|}u_s(x)\right| \leq \frac{C}{|x|}$$

when $|x| \to \infty$. Let us denote $\rho(y') = (\partial_3 u_s^+ - \partial_3 u_s^-)(y^0)$. Then (12) gives

$$u_s^\infty(\hat{x}) = \lim_{|x|\to\infty} |x|e^{-ik|x|}\langle \rho(y'), \Phi(x,y^0)\rangle_{y'}$$

should the limit exist. The distribution pairing is over $y' \in \mathbb{R}^2$. We can rewrite

$$|x|e^{-ik|x|}\langle \rho(y'), \Phi(x,y^0)\rangle = \left\langle \rho(y'), |x|e^{-ik|x|}\Phi(x,y^0) - \frac{e^{-ik\hat{x}\cdot y^0}}{4\pi}\right\rangle_{y'} + \frac{1}{4\pi}\langle \rho(y'), e^{-ik\hat{x}\cdot y^0}\rangle_{y'}.$$

We can write the C^1-test function in the first pairing on the right as

$$|x|e^{-ik|x|}\Phi(x,y^0) - \frac{e^{-ik\hat{x}\cdot y^0}}{4\pi} = \frac{e^{-ik|x|}|x|}{4\pi}\left(\frac{e^{ik|x-y^0|}}{|x-y^0|} - \frac{e^{ik|x|}}{|x|}e^{-ik\hat{x}\cdot y^0}\right)$$

which converges to zero in the C^1 topology over y', and a fortiori y^0, restricted to any compact set by Lemma 5. Please note that the C^1-seminorms are taken with respect to the y'-variable, and the absolute value makes the $e^{-ik|x|}$ that does not appear in the lemma disappear. Hence the application of the lemma is allowed. Elements of $\widetilde{H}^{-1/2}(\Omega_0)$ act well on C^1-functions, so the distribution pairing with ρ and the test function tends to zero. Thus

$$\lim_{|x|\to\infty} |x|e^{-ik|x|}\langle \rho(y'), \Phi(x,y^0)\rangle_{y'} = \frac{1}{4\pi}\left\langle \rho(y'), e^{-ik\hat{x}\cdot y^0}\right\rangle_{y'}$$

as claimed. □

3. Solving the Inverse Problem

We are ready to tackle the inverse problem in this section. In the following lemma, if ρ is integrable then $u_s^\infty(\hat{x}) = \frac{1}{4\pi}\int_{\mathbb{R}^2} e^{-ik\hat{x}\cdot y^0}\rho(y')dy'$.

Lemma 6. *Let $k \in \mathbb{R}_+$ and $\rho \in \mathscr{E}'(\mathbb{R}^2)$ be a distribution of compact support. Let*

$$u_s^\infty(\hat{x}) = \frac{1}{4\pi}\left\langle \rho, e^{-ik\hat{x}\cdot y^0}\right\rangle \tag{16}$$

for $\hat{x} \in \mathbb{S}^2$ and where the distribution pairing is over the variable $y' = (y_1, y_2) \in \mathbb{R}^2$. Then ρ is uniquely determined by u_s^∞.

Proof. The operator mapping $\rho \mapsto u_s^\infty$ is bounded and linear $\mathcal{E}'(\mathbb{R}^2) \to C^0(\mathbb{S}^2)$. This is because $\hat{x} \mapsto (y' \mapsto \exp(-ik\hat{x}\cdot y^0))$ is continuous $\mathbb{S}^2 \to \mathcal{E}(\mathbb{R}^2)$. So it is enough to show that $\rho = 0$ if $u_s^\infty = 0$. Let us assume the latter. For $\xi' \in \mathbb{R}^2$ we have

$$\hat{\rho}(\xi') = \frac{1}{2\pi}\left\langle \rho, e^{-i\xi'\cdot y'}\right\rangle$$

where the distribution pairing is over the variable $y' \in \mathbb{R}^2$. This looks similar to Formula (16) in the statement. We can rewrite

$$-ik\hat{x}\cdot y^0 = -ik(\hat{x}_1, \hat{x}_2, \hat{x}_3)\cdot(y_1, y_2, 0) = -i(k\hat{x}_1, k\hat{x}_2)\cdot(y_1, y_2).$$

Thus

$$u_s^\infty(\hat{x}) = \frac{1}{2}\hat{\rho}(k\hat{x}_1, k\hat{x}_2). \tag{17}$$

The left-hand side is zero for all $\hat{x} \in \mathbb{S}^2$. When $\hat{x}$ goes through the whole of $\mathbb{S}^2$, the sum including only two of the squares, $\hat{x}_1^2 + \hat{x}_2^2$, goes through the whole interval $(0,1)$. Alternatively

$$\hat{\rho}(\xi') = 2u_s^\infty\left(\xi_1/k, \xi_2/k, \sqrt{k^2 - \xi_1^2 + \xi_2^2}/k\right) = 0$$

for all $|\xi'| \leq k$. Since ρ has compact support, $\hat{\rho}$ can be extended to an entire function on $\mathbb{C}^2$. Since it vanishes on an open subset of $\mathbb{R}^2$ it must be the zero function. Hence $u_s^\infty = 0$ implies $\rho = 0$. □

Lemma 7. *Let $(\Delta + k^2)u_i = 0$ in $\mathbb{R}^3$. Let $\Omega \subset \mathbb{R}^3$ be a screen and u_s satisfy the direct scattering problem of Definition 2. Denote*

$$\rho(x') = \partial_3 u_s^+(x^0) - \partial_3 u_s^-(x^0)$$

for $x' \in \mathbb{R}^2$ and its properties are given in Lemma 4. If $u_i(x', x_3) \neq -u_i(x', -x_3)$ for some $x \in \mathbb{R}^3$ then

$$\overline{\Omega_0} = \operatorname{supp}\rho \tag{18}$$

for the shape Ω_0 of the screen Ω.

Proof. The function ρ is a well-defined $H^{-1/2}(\Omega_0)$-function by Lemma 4 so in particular $\operatorname{supp}\rho \subset \overline{\Omega_0}$. It remains to prove that $\overline{\Omega_0} \subset \operatorname{supp}\rho$.

Assume the contrary that $\overline{\Omega_0}$ is not contained in the support of ρ. Then neither is Ω_0 because if $\Omega_0 \subset \operatorname{supp}\rho$ then $\overline{\Omega_0} \subset \overline{\operatorname{supp}\rho} = \operatorname{supp}\rho$. Because Ω_0 is an open set and $\operatorname{supp}\rho$ is closed there is $x_0' \in \Omega_0$ and $r > 0$ such that $B(x_0', r) \subset \Omega_0 \setminus \operatorname{supp}\rho$.

Let us study the behavior of u_s in the tube $B(x_0', r) \times \mathbb{R}$. We have $\rho = 0$ on $B(x_0', r)$. Recall Formula (4), which combined with the vanishing of ρ implies that $(\Delta + k^2)u_s = 0$ in the whole tube, and interior elliptic regularity implies that u_s is smooth there. In addition the formula implies that $u_s(x_1, x_2, x_3) = u_s(x_1, x_2, -x_3)$ for all x in the tube. The vanishing of ρ gives $\partial_3 u_s^+ = \partial_3 u_s^-$ on the base of the tube. These two imply that actually $\partial_3 u_s(x', 0) = 0$ for $x' \in B(x_0', r)$.

We have the following

$$u_s = -u_i, \tag{19}$$
$$\partial_3 u_s = 0 \tag{20}$$

on $B(x_0', r) \times \{0\}$. Let us calculate the higher order derivatives. Please note that ∂_3^j and $(\Delta + k^2)$ commute, and $(\Delta + k^2)u_s = 0$ in the tube. Thus

$$0 = \partial_3^j(\Delta + k^2)u_s = (\Delta + k^2)\partial_3^j u_s = (\Delta' + k^2)\partial_3^j u_s + \partial_3^{j+2} u_s$$

in the tube, and we denote $\Delta' = \partial_1^2 + \partial_2^2$. This gives $\partial_3^{j+2} u_s = -(\Delta' + k^2)\partial_3^j u_s$. Let us restrict ourselves to $B(x_0', r) \times \{0\}$ next. By induction and (19) and (20) we see that

$$\partial_3^j u_s = \begin{cases} (-1)^{j+1}(\Delta' + k^2)^j u_i, & j \in 2\mathbb{N}, \\ 0, & j \in 2\mathbb{N}+1 \end{cases}$$

on $B(x_0', r) \times \{0\}$. This can still be simplified! Recall that u_i is an incident wave, so $(\Delta + k^2)u_i = 0$ everywhere. This means that $(\Delta' + k^2)u_i = -\partial_3^2 u_i$, and a fortiori $(\Delta' + k^2)^j u_j = (-\partial_3^2)^j u_i$ everywhere by the commutating of ∂_3^2 and $(\Delta' + k^2)$. This implies

$$\partial_3^j u_s = \begin{cases} -\partial_3^j u_i, & j \in 2\mathbb{N}, \\ 0, & j \in 2\mathbb{N}+1. \end{cases} \tag{21}$$

The other derivatives, ∂_1 and ∂_2 commute with each other and ∂_3, so finally we have

$$\partial^\alpha u_s = \begin{cases} -\partial^\alpha u_i, & \alpha_3 \in 2\mathbb{N}, \\ 0, & \alpha_3 \in 2\mathbb{N}+1 \end{cases} \tag{22}$$

on $B(x_0', r) \times \{0\}$ for all multi-indices $\alpha \in \mathbb{N}^3$.

Let us define

$$\tilde{u}_i(x) = \frac{1}{2}\big(u_i(x_1, x_2, x_3) + u_i(x_1, x_2, -x_3)\big)$$

for all $x \in \mathbb{R}^3$. This satisfies the Helmholtz equation everywhere, and is an incident wave because u_i is one. We see that

$$\partial^\alpha \tilde{u}_i(x) = \frac{1}{2}\big(\partial^\alpha u_i(x_1, x_2, x_3) + (-1)^{\alpha_3}\partial^\alpha u_i(x_1, x_2, -x_3)\big)$$

so

$$\partial^\alpha \tilde{u}_i = \begin{cases} \partial^\alpha u_i, & \alpha_3 \in 2\mathbb{N}, \\ 0, & \alpha_3 \in 2\mathbb{N}+1 \end{cases} \tag{23}$$

on $B(x_0', r) \times \{0\}$. By (22) we see immediately that $\partial^\alpha u_s = -\partial^\alpha \tilde{u}_i$ on the base of the tube for all $\alpha \in \mathbb{N}^3$. Both functions u_s and $-\tilde{u}_i$ satisfy the Helmholtz equation not only in the tube but also in $\mathbb{R}^3 \setminus \overline{B}(0, R)$, where $R > 0$ is large enough that $\overline{\Omega} \subset B(0, R)$. Solutions of the Helmholtz equation are real analytic. Because their Taylor-expansions at $(x_0', 0)$ are equal, the functions are equal in the component of $\big(B(x_0', r) \times \mathbb{R}\big) \cup \big(\mathbb{R}^3 \setminus \overline{B}(0, R)\big)$ that contains $(x_0', 0)$, so in particular $u_s = -\tilde{u}_i$ in all of $\mathbb{R}^3 \setminus \overline{B}(0, R)$.

The function u_s satisfies the Sommerfeld radiation condition, so so does $\tilde{u}_i$. On the other hand $(\Delta + k^2)\tilde{u}_i = 0$ in all of $\mathbb{R}^3$, so $\tilde{u}_i$ is the zero function (Use e.g., (9) for a large ball whose radius grows to infinity. The boundary integral decreases to zero as was seen for the first integral in (15).), which means that u_i is antisymmetric with respect to $\mathbb{R}^2 \times \{0\}$, a contradiction. Hence $\overline{\Omega_0} \subset \operatorname{supp} \rho$. $\square$

The solution to the inverse problem of determining a screen Ω from the knowledge of a single incident wave u_i and the corresponding far-field u_s^∞ scattered from the screen comes from a combination of determining ρ from the far-field, and then Ω from ρ. There is a slight surprise, namely that the problem is only solvable for incident waves that are not too (anti)symmetric. However, one sees that antisymmetry is not the deciding factor: what matters is whether u_i is identically zero on the screen. By a similar argument as that at the end of the proof of Lemma 7, we see that if $u_i = 0$ on a non-empty open subset of $\mathbb{R}^2 \times \{0\}$ then $u_i(x', x_3) = -u_i(x', -x_3)$ for all $x \in \mathbb{R}^3$. It is interesting to see that partial invisibility is achieved inside thickened screens as long as the incident plane wave comes from a direction almost parallel to the screen's normal [39]. The direction of incident waves seems very important in scattering from objects that are thin in one direction.

Proof of Theorem 2. Theorem 1 and Lemma 6 imply that $\rho = \tilde{\rho}$ when $u_s^\infty = \tilde{u}_s^\infty$. If u_i is not antisymmetric with respect to $\mathbb{R}^2 \times \{0\}$ then

$$\overline{\Omega_0} = \operatorname{supp} \rho = \operatorname{supp} \tilde{\rho} = \overline{\tilde{\Omega}_0}$$

by Lemma 7. Because Ω_0 is a smooth domain, we have $\Omega_0 = \operatorname{int} \overline{\Omega_0}$, and similarly for $\tilde{\Omega}_0$. Thus, the equation above implies $\Omega_0 = \tilde{\Omega}_0$ and by lifting, $\Omega = \tilde{\Omega}$.

If u_i is antisymmetric then $u_i = 0$ everywhere on $\mathbb{R}^2 \times \{0\}$ and $u_s = 0$ satisfies all conditions of the direct scattering problem. Since solutions to the direct scattering problem (2) are unique by ([37] [Thms 2.5–2.7]), this is the only solution. Thus, $u_s = \tilde{u}_s = 0$ and the same holds for their far-fields. This is irrespective of the shape of $\Omega, \tilde{\Omega} \subset \mathbb{R}^2$. $\square$

Author Contributions: All three authors worked equally on the article. L.P. and S.S. initiated the project and wrote the first pen and paper version of it. E.B. joined the project to solve a few remaining mathematical issues and format the manuscript and finalize it into publishable form. All authors have read and agreed to the published version of the manuscript

Funding: The research of the authors is supported by Estonian Research Council grant PRG832.

Acknowledgments: We thank Markku Lehtinen for useful discussions.

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

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